E; join BC and produce it to cut the side OY at F; join EF. Join DC and produce it to cut EF in G; join GA and produce it to cut OY in H. It is easy to see that H is the point required.

Pendulum Clocks and their Errors.

By H. R. A. MALLOCK, F.R.S.

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The measurement of time by pendulum clocks is one of the most accurate measures in daily use, but even with the best existing clocks it is not uncommon to find that the rate may vary from time to time by as much as half a second a day, roughly about 1 part in 150,000.

Clocks are rated by comparison with the rotation of the earth, which is assumed to be constant, and it is a matter of some interest to inquire into the possibility of making a clock good enough to rate the earth by.

The gradual reduction of the angular velocity of the earth as deduced from astronomical data used to be taken as about 22 seconds per century,* but I believe that this estimate is now considered too high.†

At any rate, if a clock is to be accurate enough to give a measure of the change in the earth's angular velocity, it must have a variation of rate not much exceeding one second per century, or about 1 part in 3,000,000,000. It seems hardly possible that this order of accuracy should be reached by any mechanical appliance.

In the following notes, however, I have stated the various causes which may bring about a change in the rate of a pendulum, and the permissible variation of the different arguments involved if the variation of rate due to each of them is not to exceed 1 part in 100,000,000.

The causes of variation of rate may be grouped under three heads, namely :—

(*a*) Such as would affect a pendulum swinging freely *in vacuo*.

(*b*) Variations due to the action of the atmosphere on a free pendulum.

(*c*) Variations due to the action of the mechanism which maintains the vibration.

Of these it will be seen that (*c*) is the most important and the most

* Thomson and Tait, vol. 1, Part I, p. 400.

† Variation of the length of the day due to contraction by secular cooling would not be indicated by a pendulum clock.

difficult to deal with. In the early days of clock making, before the approximate isochronism of pendulums was properly appreciated, more attention was given to the mechanism than to the escapement, the latter being looked on merely as a sort of brake to prevent the weights running down too fast, but for the last 150 years at least it has been recognised that the function of the clock mechanism was, in the first place, to maintain the swing of the pendulum with as little interference as possible, and, secondly, to count the swings.

In what follows the variations of rate are expressed in fractions of the natural period of the pendulum, and the integral error in the recorded time over any stated period is the same fraction of that period, and is independent of the actual length or natural period of the pendulum.

Beginning with the free, simple pendulum oscillating *in vacuo* with vanishing amplitude; if we take a simple pendulum of length λ, for which $\tau = 2\pi\sqrt{\lambda/g}$ for infinitesimal arcs, then

$$(1)\quad d\tau/d\lambda = \tau/2\lambda, \qquad \text{also} \qquad (2)\quad d\tau/dg = \tau/2g.$$

If we take a pendulum with a bob whose moment of inertia is $M\kappa^2$, $\tau = 2\pi\sqrt{(h^2+\kappa^2)/gh}$ (where h is the length of the pendulum rod measured from the support to the centre of inertia), we have $d\tau/d\kappa = 2\pi\kappa/h\sqrt{gh}$ or $\tau\kappa/h^2$.

(4) A pendulum is only approximately isochronous, and in this respect is at a disadvantage compared with a balance wheel stabilised by a spring.

Keeping only the first terms of the series, expressing the period τ_α in an arc whose circular measure is α, in terms of α and τ_0 (the period for infinitely small arcs), if α is small, $\tau_\alpha = \tau_0(1+\alpha^2/16)$ and $d\tau/d\alpha = \alpha/8$.

(5) Pendulums are generally suspended by a very thin spring, but knife-edges have also been used, and in estimating the equivalent length (λ) corrections are required in both cases.

First as regards spring suspension. The spring is clamped at the upper end and clamped also to the pendulum rod. It can be shown that the equation to the curve into which the spring is bent is (for small pendulum arcs) $y = y_1 e^{-x/p}$. In this equation x is measured along the virtual line of suspension of the pendulum, the axis of Y being the normal to X passing through the point of suspension (see fig. 1), and $p = \sqrt{A/W}$, where W is the weight of the pendulum and A depends on the elastic constant of the spring and its form. For a strip of width b and thickness $2c$, $A = 2nbc^3$ (n being Young's modulus for the material).

The virtual length of the pendulum is shortened by $\sqrt{A/W}$.

(6) If the suspension is by a knife-edge the correction is of a different

kind. A knife-edge must be regarded as a part of a cylinder of small radius (r) which rolls on the supporting surface. If the pendulum oscillated on the knife-edge as a pivot (*i.e.* if it turned without rolling) the length of the pendulum measured from the axis to the centre of oscillation would be the length of the equivalent simple pendulum. If, as is or always should be the case, the knife-edge rolls on its support, λ is shorter by r than the measured length.

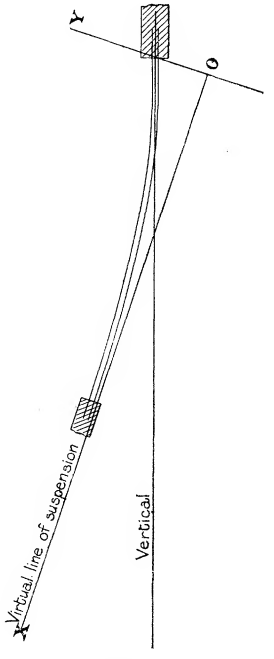

FIG. 1.

It is rather a nice question to determine the best value for r. The smallest value which it can have is determined by the strength and hardness of the material, and if the edge is made too sharp to begin with it will be crushed or deformed in use until either the bearing surface is sufficient to take the stress or (if the support is softer than the knife-edge) it has to some extent embedded itself. In the latter case an angular motion about the axis would involve a certain amount of sliding.

It is not at all an easy matter to work a knife-edge to a truly cylindrical form, and the difficulty increases as the radius diminishes. A very common feature in knife-edges is that of having a small almost flat surface along what would naturally be considered as the edge, and, in fact, it is difficult to handle or place two very hard surfaces in contact without leaving some trace of impact on either.

(7) Suppose that the edge is flattened for a distance a' on each side of the centre line, and that $a' = r\alpha'$, then if $\lambda\alpha$ is the linear amplitude of the vibration of the pendulum, its period will be $\tau_\alpha\left(1-\frac{2}{\pi}\sin^{-1}\frac{r\alpha'}{\lambda\alpha}\right)$, as may be seen from fig. 2. Thus the period diminishes indefinitely with the arc. The motion changes its character when $\lambda\alpha < r\alpha'$.

(8) Another cause which will make the actual period of a given pendulum differ from τ_α is the yielding of supports.

The lateral force exerted by the pendulum on its supports must cause some yielding, and the effect of this on the period of the pendulum may be very simply stated as that of an addition to λ of the length of a pendulum whose period is the natural period of the support when loaded with a mass equal to that of the pendulum.

(9) The length of a pendulum varies during the swing in virtue of the

stretching of the rod by centrifugal force. The maximum extension due to this is $4W\pi^2\lambda a/nsg\tau^2$, where a is the linear amplitude of the oscillation, s the sectional area of the pendulum rod, and n Young's modulus for the material. The alteration of period is that due to lengthening the pendulum by half this quantity.

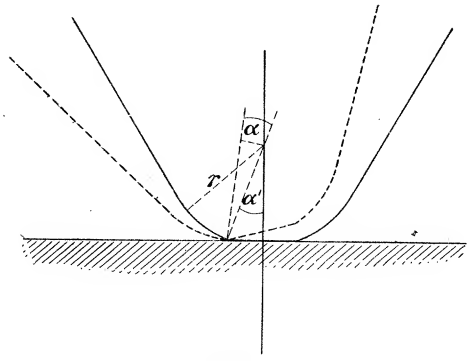

FIG. 2.

(10) In many pendulums mercury is used in the bob for the purpose of compensating temperature effects. The moment of inertia of such a bob differs from that of a solid of the same form, and if the mercury behaved as a perfect fluid the difference would be a function of the shape, and has been calculated for some simple forms.*

When, however, the viscosity of the mercury is taken into account, not only the shape but the absolute period enters into the expression for the moment of inertia in virtue of the rate of propagation of viscous disturbance in the fluid.† In consequence, when the period is very long the liquid bob tends to have the same moment of inertia as a solid of the same form, and when the period is very short the moment of inertia approximates to what it would be if viscosity did not act.

(11) When an oscillation of any kind is opposed by a force proportional to the velocity, and when that force is small compared to the positional force, the alteration of the period of the resisted oscillation is of the second order, but cannot on that account be neglected in the case of pendulums used for the reckoning of time.

The relation between the period τ_α for an unresisted motion of amplitude α, and the actual period τ_α', when resisted, is $\tau_\alpha' = 2\pi\tau_\alpha/\sqrt{(4\pi^2+\beta^2)}$,‡ where

* See Stokes, 'Collected Papers,' vol. 1, p. 188.

† See Lamb's 'Hydrodynamics,' § 328, p. 328.

‡ See Maxwell's 'Electricity,' vol. 2, p. 338, from which this result may be easily deduced.

β is the angle of the equiangular spiral of which the motion of the resisted oscillation may be taken as the projection. Or, in other words, if $\Delta\alpha$ ($= 2\pi\alpha\beta$) is the diminution of amplitude which occurs in each complete oscillation when the arc is α, then, owing to the extinction, the period is diminished by $\tau_a[1 - 2\pi/\sqrt{(4\pi^2 + \beta^2)}]$, which, when β is small, becomes $\tau_a\beta^2/8\pi^2$.

(12) The effect of a constant resistance on an oscillation is to make each successive arc less than the preceding by a constant quantity. The influence on the period may be gathered from the fact that each time the direction of motion changes sign the zero about which the oscillation takes place is shifted in a direction opposite to that of the motion, and by a constant quantity proportional to the resistance. In fig. 3 let ABC represent the

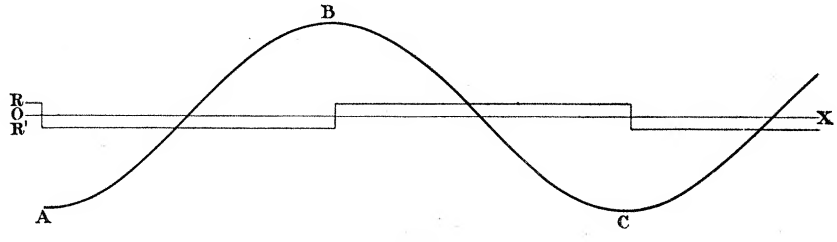

FIG. 3.

unresisted oscillation. Then if OR represent the resistance (on such a scale that the ordinates of ABC represent the positional forces), the retarded oscillation for the half period from A to B will be performed as an unresisted oscillation about the zero axis $y = \mathrm{OR}$, and from B to C as an unresisted oscillation about $y = -\mathrm{OR}$, and the arc α is reduced by OR/λ at each half swing.

Thus, if the time is measured between successive "elongations," the period is unaltered by constant friction; but, between successive crossings of the unresisted zero line, each successive half period is longer than the preceding one by $\frac{\tau}{2\pi}\left(\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n}\right)$. When the sum of these excesses reaches the value $\tau/4$, the opposing frictional force is equal to the positional force and the motion ceases.

This proposition is useful in some cases, for it shows that the free period of a pendulum resisted by constant friction can be found, very nearly, by counting the swings until the motion stops, then if the number counted is N, and the whole time is T, the free period is $4\mathrm{T}/(4\mathrm{N}+1)$.

The various causes enumerated above constitute, I believe, a complete list of sources of variation of period which may affect the motion of an unmaintained oscillation *in vacuo* by a quantity exceeding $\tau \times 10^{-8}$.

If the object of the pendulum observation were to connect λ and g, that is, to deduce a standard of length by gravity experiments, or *vice versâ*, all matters which might affect λ would have to be taken into account, but if the pendulum is merely used as a time-keeper, the variations of λ only, and not the absolute values of h or κ, etc., are of any importance.

Action of the Surrounding Gas.

(13) The air or gas in which the pendulum swings affects the motion in four ways. In the first place, it tends to float the pendulum, and in effect reduces the force of gravity in the ratio of the weight of the displaced air to the weight of the pendulum.

(14) Secondly, assuming for the moment that the air is devoid of viscosity and flows round the pendulum according to the "electric" law, the mass in motion is not only that of the pendulum itself, but it involves also the mass of the disturbed air. This extra mass can be stated as a certain fraction of the mass of a volume of air equal to the volume of the pendulum. The fraction depends on the form of the pendulum (for a cylinder it is unity, and for a sphere $\frac{1}{2}$).

This extra mass, which does not contribute to the positional force, is equivalent to a further reduction of the action of gravity.

(15) Thirdly, when viscosity is taken into account, it is found that, in addition to the mass set in motion by stream-line action, a certain quantity of air is dragged after the pendulum, and that this mass (which also contributes nothing to the positional force) counts as a further reduction in the intensity of gravity.

(16) This, however, is not the only effect which viscosity has, for the motion of the air dragged or rubbed along is not in quite the same phase as the pendulum itself, that is, part of the motion is proportional to the acceleration of the pendulum and part to its velocity.

The latter gives rise to extinction, and alters the period, as mentioned in (11). The investigation of these subjects is due to Stokes.* There is no reason to doubt that the results there given apply to the cases dealt with with sufficient accuracy as far as regards the flotation and stream-line effects, but the treatment of viscosity is more open to question when the velocities concerned are as large as those of clock pendulums. These velocities, indeed, are small in the ordinary sense, but are often sufficient to initiate eddies in the air—an action not taken into account in the calculations.

* 'Collected Papers,' vol. 3, or 'Camb. Phil. Soc. Trans.,' vol. 9.

Action of the Maintaining Mechanism.

To compensate for the gradual loss of energy which occurs in the oscillations of all real systems, some means must be provided to maintain the motion. This is done by the escapement, which in general consists of an arrangement of wheels and levers driven by a spring or by weights, but whose motion is intermittent, and so controlled by the pendulum itself, that during part of the swing the driving force acts on the pendulum in the direction of its motion.

The mechanical connections between the pendulum and the escapement in most cases give rise to a much greater loss of energy than the free pendulum would suffer from such causes as yielding of its supports and the viscosity of the air. Hence the greater part of the work done by the driving weights goes in making up the loss of which they themselves are the cause.

The desirable qualities in an escapement are that its action should have little effect on the natural period of the pendulum, and, which is more important, that such changes of period as are produced should be as independent as possible of variations of driving force and of the coefficients of friction of the sliding surfaces.

A great number of forms have been given to escapements for pendulum clocks, but the only ones which need be referred to here are the "dead-beat" and the "gravity," since these two have practically superseded all others for accurate time-keeping.

Dead-beat Escapement.

(17) The dead-beat escapement consists of two arcs (fig. 4), AA′ and BB′, of a cylinder whose axis (O) is coincident with the axis of the pendulum. The cylindrical arcs are bounded by the two faces AC, BD, which make the constant angle β with the radius of the cylinder. The radius of the escape wheel and the radius of the cylinder are so taken that OAE and OBE are right angles, and the spacing of the teeth on the escape wheel is such that the distance AB is rather greater than a whole number + one-half times the interval between the successive teeth. Suppose, now, the pendulum to be about to pass through the vertical from right to left; at this moment the tooth on the A side passes from the cylindrical arc A′A and slides over AC, accelerating the motion of the pendulum in the process. After the A tooth has passed, the escape wheel moves round rapidly under the driving force of the clock mechanism, until the B tooth is stopped by the cylindrical arc BB′. Against this arc it rests until the pendulum returns from left to right. The tooth then slides over BD, and again accelerates motion of the pendulum, and so on.

If μ is the coefficient of friction, and f the force which the tooth of the escape wheel exerts in the direction of its motion, the force normal to the pallet surface is $f/(\mu+\tan\beta)$, and if OA = OB = r, the accelerating force on the pendulum bob is $fr/\lambda\,(\mu+\tan\beta)$, and this force lasts while the pendulum

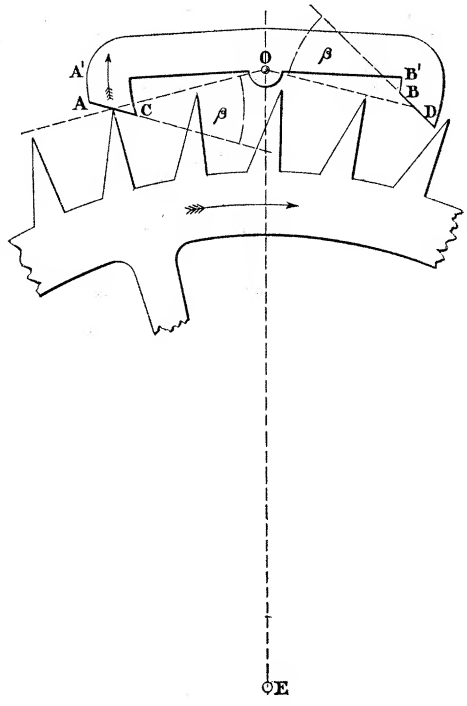

FIG. 4.

oscillates through an angle $\sigma\sin\beta/r$, where σ is the length AC or BD. Except at these times the pendulum is subjected to a retarding force $\mu fr/\lambda$, from the friction of the tooth on the cylindrical arcs AA′ or BB′.

The effect of the escapement forces on the pendulum can be represented graphically.

In fig. 5 let time be measured along OX, and let ABC represent the oscillation of the pendulum ($y = a\sin 2\pi t/\tau$ approximately, where a is the maintained amplitude). Let $y = \pm$ OG and $y = \pm$ OH be straight lines parallel to OX distant κ and e respectively from OX, where $\kappa = fr/\lambda\,(\mu + \tan\beta)$ and $e = \mu fr/\lambda$. If the scale on which ABC is drawn is such that ordinates represent the gravitational force of restitution as well as the displacement, these lines, as will be readily seen, are the equilibrium positions of the pendulum when acted on by the constant horizontal forces $fr/\lambda(\mu + \tan\beta)$ and $\mu fr/\lambda$, and the oscillation, during the

intervals for which each force lasts, is in reality a harmonic vibration of period τ_a, but whose virtual amplitude and phase change abruptly with the alteration of the zero position. Thus at P_1 the force changes suddenly

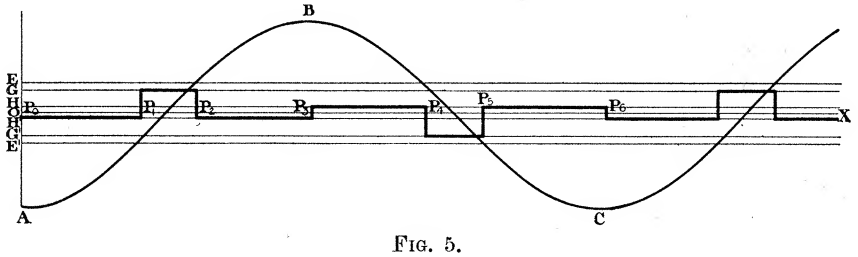

FIG. 5.

from $y - e$ to $y + \kappa$, at P_2 from $y + \kappa$ to $y + e$, at P_3 from $y + e$ to $y - e$ (on account of the change in direction of the motion), and so on. These changes are indicated by the thickened line.

Of course there is no sudden change either in the velocity or position of the pendulum when the zero positions change, and these two conditions, together with the constancy of the period τ_a for all the portions comprised between P_1P_2, P_2P_3, etc., of the oscillation, suggest a simple construction for evaluating the effect of the applied forces on the time which elapses between the successive "elongations," that is on the effective period of the pendulum.

In fig. 6 let

$$\left.\begin{matrix} OE \\ OE' \end{matrix}\right\} = \pm h, \qquad \left.\begin{matrix} OG \\ OG' \end{matrix}\right\} = \pm \kappa, \qquad \left.\begin{matrix} OH \\ OH' \end{matrix}\right\} = \pm e,$$

κ and e having the same meanings as in fig. 5.

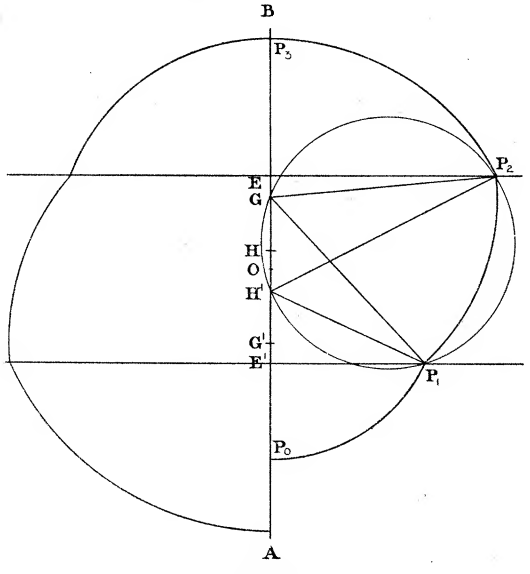

FIG. 6.

We may imagine the actual motion of the pendulum to be represented by the projection on straight line through AA′ of a point P moving with the constant angular velocity $2\pi/\tau_a$ about the centres H′, G, H′, H, G′, H, in succession. Starting from P_0 the projected motion is resisted by the constant friction e from P_0 to P_1, therefore the point moves in a circle of radius $H'P_0$ about H′; arrived at P_1, the projected motion is accelerated by the force κ due to the action of the tooth of the escape wheel on the pallets, and the point moves in a circle about G from P_1 to P_2; from P_2 to P_3 the motion is again retarded by a force e, and takes place about H′. It is unnecessary to follow the path in detail through P_4, P_5, and P_6.

Let $$P_0HP_1 = \phi_1, \quad P_1GP_2 = \phi_2, \quad P_2H'P_3 = \phi_3;$$

also let $$P_1H'P_2 = \phi'_2.$$

The construction shows that $\phi_1 + \phi'_2 + \phi_3 = \pi$, and if ϕ_2 were equal to ϕ'_2 we should have $\phi_1 + \phi_2 + \phi_3 = \pi$, so that the time taken in describing these three arcs with the angular velocity $2\pi/\tau$ would be $\tau/2$. In this case the applied forces would have made no difference in the period of the pendulum.

The equality between ϕ_2 and ϕ'_2 does not really exist except for certain values of κ and e, although the difference is of the second order.

Draw a circle circumscribing P_1, G, P_2. This circle will, in general, pass close to H′ (if it passes through H′, $\phi_2 = \phi'_2$).

It can be shown that $\phi_2 - \phi'_2$ is nearly equal to the distance (z) (perpendicular to AB) of H′ from the circumscribing circle $\times \sin\phi_1/H'P_2 \sin\phi_2$ (or $\times \sin\phi_2/HP_1 \sin\phi_1$).

In all real cases $\sin\phi_1$, $\sin\phi_2$, $H'P_2$ and HP_1, are so nearly the same that $\phi_2 - \phi'_2$ may be taken as $= z/a$, and the difference in the time of oscillation due to the action of the maintaining and frictional forces is $\tau z/2\pi a$.

As regards the alteration of phase in the different parts of the path; there is a retardation equal to GP_1H' at P_1, and a nearly equal acceleration GP_2H', at P_2. No change of phase occurs at P_0 or P_3.

Supposing that the only dissipative force acting is due to the escapement friction, and that $H'P_1 = a_1$, $GP_1 = a_2$, and $H'P_2 = a_3$, we have $a_2 = a_1 + (\kappa + e)\cos\phi_1$, and $a_3 = a_2 + (\kappa + e)\cos(\phi_1 + \phi_2)$. Now since in real cases both κ and e are so small compared to a that $\frac{1}{2}(\pi - \phi_2) = \phi_1$ and $\frac{1}{2}(\pi + \phi_2) = \phi_1 + \phi_2$, we have $\cos(\phi_1 + \phi_2) = \cos\phi_1$ nearly, so that $a_3 = a_1 + 2(\kappa + e)\cos\phi_1$. If the original amplitude at P_0 measured from O is a_0, $a_0 = a_1 + e$, and the amplitude at P_3 is $a_3 - e$.

Hence, when the escapement has been in action long enough for the motion

to have become steady, that is, as soon as the frictional loss is just balanced by work done in maintaining the vibration $a_0 = a_3 - e$; therefore

$$\cos \phi_1 = e/(k+e).$$

If the natural loss of amplitude apart from escapement friction is e', we have

$$a_3 = a_1 + 2(\kappa+e)\cos\phi_1 + e' \quad \text{and} \quad \cos\phi_1 = (e+e')/(\kappa+e);$$

and since also $\cos\phi_1 = h/a_0$ nearly,

$$a_0 = h(\kappa+e)/(e+e').$$

This gives the amplitude of the swing in terms of h (the distance on either side of OX (fig. 5) at which the pallets come into action), and of the applied forces other than gravity.

The effect of these forces on the period, when the amplitude remains constant, is very small (of the order less than 10^{-8}), but since the forces represented by κ and e depend on the coefficient of friction between the teeth of the escape wheel and the pallets, a_0 and therefore the period is affected by any change in this coefficient, even if the driving force, f, is constant.

In ordinary clocks, however, in which the escape wheel is driven through a train of wheel work, f itself is subject to variations if, through dirt thickening of oil, or any other cause, the efficiency of the transmission varies.

The investigation just given shows that with a "dead-beat" escapement errors of rate depend on the induced variations of arc of vibration, and not, to any appreciable degree, on the difference of period, which variable friction might be supposed to make in an oscillation of fixed amplitude, and also that the extreme smallness of this direct effect is due to the fact that the maintaining force is applied symmetrically, and lasts for the same time before and after the pendulum passes through the vertical.

Gravity Escapement.

(18) This is made in several forms, but the principle is that the mechanism of the clock alternately raises two small weights, one on either side of the pendulum; these are retained in the raised position until released by the latter some time before it reaches its extreme elongation. The released weight then moves with, and accelerates the motion of, the pendulum until it again passes through the vertical. After this the pendulum is free until it releases the weight on the opposite side, which then moves with the pendulum as did the first, and so on.

While the weights act, the pendulum is virtually oscillating about a zero

distant $\lambda w/W$ from the vertical; w and W being the weight released by the escapement and the weight of the pendulum respectively.

Figs. 7 and 8 give a diagrammatic picture of the applied forces.

In fig. 7, ABC is the path of the pendulum; the lines through E, E′, F,

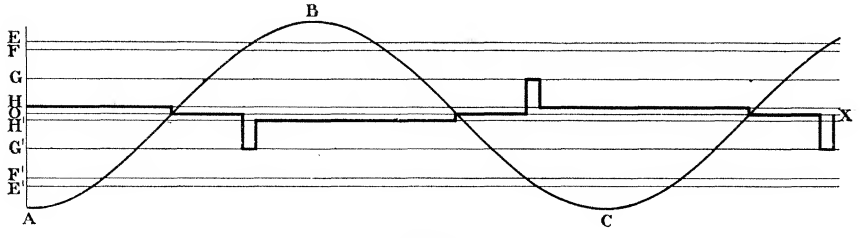

Fig. 7.

and F′ give the distance from the vertical at which the pendulum begins and completes the release of the weights. G and G′ represent the alteration of pendulum zero which occurs while the release is being accomplished, H and H′ in the same way represent the zero alteration due to the action of the weights. The thickened line indicates the sequence in time of the applied forces.

Supposing, as in the case of the "dead-beat," that the motion of the pendulum is represented by the projection of a point moving with uniform angular velocity about the successive zero positions, in fig. 8 (the lettering

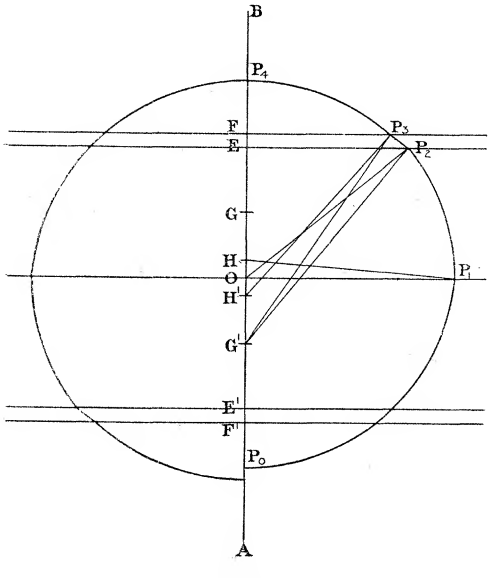

Fig. 8.

being the same as in fig. 7) let $OE = OE' = h$; $EF = E'F' = \delta h$; $OG = OG' = \kappa$; $OH = OH' = e$. Starting from P_0, let the initial amplitude $OP_0 = a_0$, and let $HP_0 = HP_1 = a_1$, $OP_1 = OP_2 = a_2$, $GP_2 = GP_3 = a_3$ $H'P_2 = P'P_4 = a_4$, and let $OP_4 = a_\pi$, *i.e.* the amplitude after the first half oscillation.

Let $P_0HP_1 = \phi_1$, $P_1OP_2 = \phi_2$, $P_2G'P_3 = \phi_3$, $P_3H'P_4 = \phi_4$.

We have to find a_π and $\phi_1+\phi_2+\phi_3+\phi_4$ in terms of a_0, h, δh, κ, and e; and if we assume, as is true in real cases, that κ and e are small compared to h (and of course to a_1) we have $a_1 = a_0+e$, $a_2 = a_1$, $a_3 = a_1+\kappa h/a$, $a_4 = a_3-(\kappa-e)h/a$, also $\phi_1 = \pi/2-e/a$, $\phi_3 = \delta h/\sqrt{(a^2-h^2)}$, and $P_2OP_3 = \phi_3 a_3/a_4$.

Now
$$e/a+\phi_1+\phi_2-P_2OP_3+\phi_4 = \pi,$$
or
$$\phi_1+\phi_2-\phi_3 a_3/a_4+\phi_4 = \pi-e/a.$$
Thus
$$\phi_1+\phi_2+\phi_3+\phi_4 = \pi-e/a-\phi_3(1-a_3/a_4).$$

From these equations we find, approximately,
$$a_\pi = [a_0^2+e(a_0+h)]/(a_0+e)$$
and
$$\pi-(\phi_1+\phi_2+\phi_3+\phi_4) = \frac{e}{a+e}+\delta h\frac{h\kappa}{\sqrt{a^2-h^2}(a^2+eh)}.$$

The gain in amplitude $(a_\pi-a_0)$ therefore is $eh/(a+e)$ and the time of the oscillation is less than τ_a by
$$\frac{\tau}{2\pi}\left(\frac{e}{a+e}+\delta h\frac{h\kappa}{\sqrt{a^2-h^2}(a^2+eh)}\right).$$

The second term in the bracket is the smallest, but it is also the most variable, for it is affected not only by the variation of the coefficient of friction during the release of the weights, but also by any change in the efficiency of the clock train which alters the force on the detent upholding the weights.

The amplitude of the oscillation with this form of escapement can be found by equating $eh/(a+e)$ to the natural decrement of the arc per half period, and since this does not involve κ (except in the second order of small quantities) the amplitude ought to remain very constant, but the alteration of period due to the second term may be appreciable.

Both the dead-beat and gravity escapements owe such small variations in time-keeping as they are subject to, directly or indirectly, to variations in the driving force.

(19) In an escapement of which a model is now shown* this source of error

* The model referred to was exhibited at the May Soirée, 1911, but (owing to the limited time at disposal) not at the reading of the paper.

does not exist, and it is hoped that with it variations of rate may be of a smaller order.

The pendulum carries a cross arm AB slightly above the level of the axis (fig. 9). From this arm three fine wires, C_1, C_2, C_3, one in the centre and one at each end, project downwards and terminate in a line passing at right angles through the axis of suspension. The centre wire dips always in a mercury cup D_2, and the end wires dip alternately in mercury cups D_1 and D_3, whose level is so adjusted that contact takes place near the end of each swing. The three wires are electrically connected, and hence a current can be transmitted alternately through the end cups to an electro-magnet.

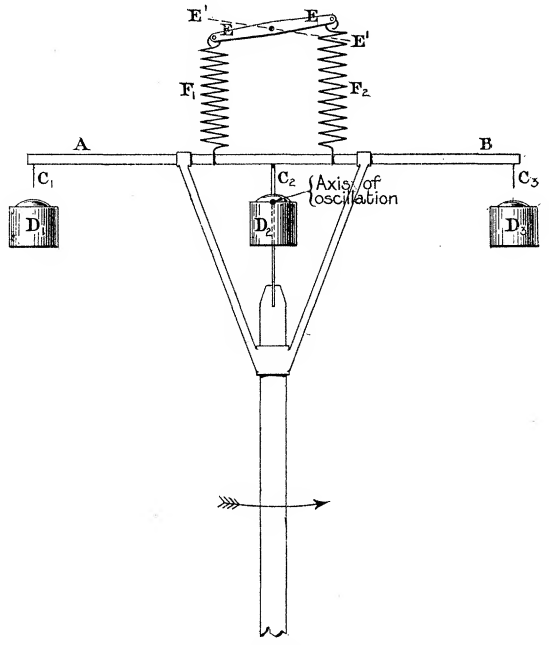

FIG 9.

In the present model the electro-magnet, through a kind of remontoir, works a short lever E whose axis is parallel to, and some distance vertically above, the axis of suspension. This lever can assume either of the two positions (E) or (E'), and takes these positions very rapidly when the current passes. At each end of the lever very fine spiral springs, F_1, F_2, descend and are connected below to the cross arm on the pendulum.

The forces acting on the pendulum are shown in fig. 10 by the thickened line.

Here E and E′ are the distances of the pendulum from the vertical at which the electric contacts are made. F and F′ are the changes of zero due to

change of the mean force exerted by the springs when the lever shifts from one position to the other.

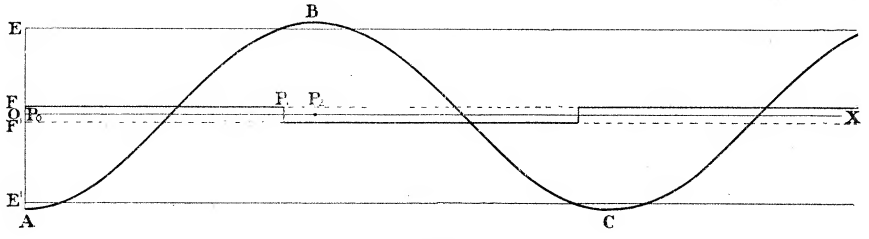

FIG. 10.

In fig. 11 let

$OP_0 = a_0$, $FP_0 = FP_1 = a_1$, $F'P_1 = F'P_2 = a_2$, $P_0FP_1 = \phi_1$, $P_1F'P_2 = \phi_2$,

then $$\phi_1 + \phi_2 = \pi - FP_1F'.$$

Let $$OE = OE' = h \quad \text{and} \quad OF = OF' = e,$$

then $FP_1F' = 2e\sqrt{(a^2-h^2)}/a$ and $a_2 = a_1 + 2eh/a_2 = a_0 + 4eh/a$.

Hence $$a_\pi = a_2 - e = a_0 + 2eh/a_0$$

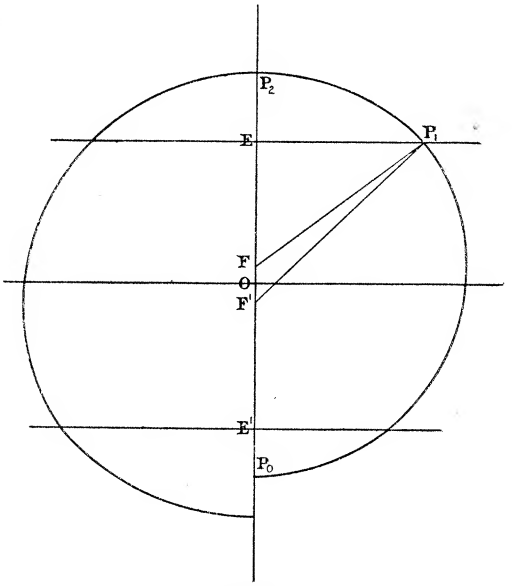

FIG. 11.

Thus, each half swing of the pendulum is quickened by $\tau e\sqrt{(a^2-h^2)}/\pi a^2$ and the amplitude (if there is no dissipation) at the same time increased by $2eh/a$.

The natural period of the pendulum is quickened by the action of the springs, whether the lever moves or not. Let L_0 be the unextended length of

the springs and let the extension of both, when the lever is in its middle position, be L_1, also let w be the weight which will extend the springs to twice their natural length. Then if w is equal to the weight of the pendulum $\times q$, and r is the radius of the lever, the period, under the action of both gravity and the two springs, is $2\pi\sqrt{[\lambda/g(1+qL_1r/L_0\lambda)]}$.

The value of e in terms of L_0, q, r, and the extension ΔL of the springs caused by the movement of the lever from one extreme position to the other is $g\Delta Lrq/L_0\lambda$.

Thus the natural period of the pendulum is $2\pi\sqrt{[\lambda/(g+e\Delta L/L_0\lambda)]}$ and the alteration of this period due to the maintaining action of the escapement is $\tau e\sqrt{(a^2-h^2)}/\pi a^2$.

I have had no extended experience yet with this escapement. Its use in practice will depend on whether the mercury contacts can be made reliable.

Of the three escapements which have been considered the dead-beat is the only one in which there is no alteration of period except that depending on change of amplitude, but the amplitude changes quickly with change in the driving force, and the latter is liable to variations from causes which cannot altogether be avoided or eliminated.

In the gravity escapement if properly made the maintaining force is very constant and nearly independent of friction, but a change in the coefficient of friction, or of the pressure between the parts which have to slide while the weight is being released, may make a very appreciable difference in the rate.

In the escapement shown in the model the maintaining force is independent of friction,* but subject to a small change depending on the variation of the elastic coefficient of the springs with temperature. But if the material of the spring is properly chosen this is not important, especially as the spring force itself in this escapement is much smaller than the requisite maintaining force for the other two, because the only work to be done is to make up the natural loss of arc due to air resistance and yielding of the supports.

In order to show the order of errors of period due to the various sources mentioned, I subjoin examples computed for a "seconds" pendulum (*i.e.*, a pendulum with 2 seconds period, whose length is a little more than 39 inches) with a bob weighing 10 lbs. (corresponding to $21\frac{1}{2}$ cubic inches of mercury or 23 cubic inches of lead nearly). The other data, which are such as may be met with in actual clocks, are given as required.

(1) and (2) If the period is to remain constant to within $\tau\times10^{-8}$ the length

* Except in so far as this may influence the time taken to shift the lever from one position to the other and thus to alter the phase at which the maintaining force is applied.

must not vary by more than $7{\cdot}5 \times 10^{-7}$ inches or the effective gravitational force by more than $6{\cdot}6 \times 10^{-7}$ foot-seconds^2.

In connection with possible variation of the force of gravity it may be noticed that if the value of gravity at the earth's surface is increased in virtue of a general contraction of the earth, the length of the day, as measured by the pendulum, would remain unaltered, because the speed of the earth's rotation would be thereby increased in the same ratio as the frequency of the pendulum; and it is only by a comparison with some period unaffected by gravity (*e.g.*, the period of a mass on a spring) that the change could be recognised. A change of 10^{-8} in the absolute period of a pendulum would be produced by a shrinkage of only 2 inches in the radius of the earth.

(3) The radius of gyration of the bob must not vary more than $7{\cdot}9 \times 10^{-6}$ inch.

(4) If the arc of vibration is of 1° it must not vary by more than 1″ if the same constancy of period is to be maintained.

(5) The action of the suspension spring is twofold. In the first place it increases the stabilising force and thus acts to shorten the period, and secondly the effective length of the pendulum is shortened by $2nbc^3/3W$.

To get a numerical value for the shortening, assume that the suspension spring is subjected to a longitudinal stress of 20 tons per square inch by the weight of the pendulum, and that its width is $\frac{1}{2}$ inch. This gives 0·000334 inch for c, and we may take for steel $n = 3 \times 10^7$ lbs. per square inch.

With these data $\sqrt{(2nbc^3/3W)} = 0{\cdot}0104$ inch, which corresponds to a shortening of the period by 1 part in 7500.

(6) If a knife-edge suspension is used, the probable value of the radius may be anything from 0·003 or 0·002 inch down to the limiting value given by the strength of the material, and this radius has to be deducted from the effective length of the pendulum.

(7) If the knife-edge terminates in a flat surface 0·001 inch in width, the period of the pendulum whose arc of vibration is 1° is shortened by $\tau \times 4{\cdot}55 \times 10^{-6}$, and the smaller the arc assumed the greater will be the shortening of the period due to the flat.

The difficulty of ensuring a good cylindrical surface is a strong objection to the use of knife-edges.

(8) The stiffness of any given support can generally be found by tapping, and thus determining the pitch of the note given out by the natural vibration, care being taken that the natural vibration so caused is of the same type as the forced vibration which the oscillation of the pendulum would set up.

For the ordinary wooden case in which many standard clocks are

mounted, this natural period may be as great or even greater than 1/50 second. When the pendulum is mounted on strong metal brackets fixed to masonry the period may be as small as 1/1000 of a second.

The corrections to the effective length in the two cases are 0·0039 inch for the period of 1/50 second and 0·000001 inch for the period of 1/1000 second.

On a seconds pendulum these would correspond to a difference of 5×10^{-5} and $1{\cdot}3 \times 10^{-8}$ seconds respectively between the actual period and the period of the same pendulum if rigidly supported.

(9) The effect of centrifugal force in altering the mean length of the pendulum may be just sensible.

We have for the maximum extension of the rod $4\pi^2\lambda a/nsgt^2$, and assuming that the suspension rod is steel and has a sectional area of 1/16 square inch, the other quantities remaining as before, viz., W = 10 lbs., $\lambda = 39$ inches, and arc of swing 1° (so that the linear amplitude $a = 0{\cdot}68$ inch), the maximum variation of length is $3{\cdot}62 \times 10^{-7}$ inch, but the effect on the period, viz., $\tau \times$ half extension$/2\lambda$, is less than $\tau \times 10^{-8}$.

(10) For the calculations referring to the difference between the moment of inertia of a solid cylinder and an equal and similar volume of liquid, see Stokes, "Some Cases of Fluid Motion."*

With regard to the effect of temperature on the viscosity of mercury, I have no data. In any case, judging from Stokes' experiments and their difference from his calculated results, both points should be determined by suitable experiments rather than by a theory which does not take account of surface friction.

(11) As an example of the variation of period due to resistance varying as the velocity, suppose the natural decrease of arc to be such that the loss is 1/1000 part of the arc per swing, that is, the angle $\beta = 1/2000\,\pi$, then $\tau\beta^2/8\pi^2 = \tau \times 3{\cdot}2 \times 10^{-9}$, and is therefore negligible.

(13) (14) According to the investigations of Stokes† the sum of the effects of the buoyancy and inertia of the air on the period is equivalent to an increase of the mass of the pendulum, but not of its weight. For a cylindrical pendulum bob this increase is twice, and for a spherical bob one and a-half times, the mass of the air displaced, the extra mass being supposed concentrated at the centre of inertia, and not subject to gravitative attraction. That is to say, W = (not Mg, but) $(M + c\rho V)g$. In effect this is equivalent to diminishing the effective force of gravity by $c\rho V$ or (if $\rho V = qM$) by cq. If we take the density of the air as 1/800 and that

* 'Collected Papers,' vol. 1, p. 64, and the Supplement thereto, *ibid.*, p. 189; or 'Camb. Phil. Soc. Trans.,' vol. 8.

† 'Collected Papers,' vol. 3.

of the pendulum as 10, $q = 1/8000$. For a cylinder $c = 2$, and the alteration of period due to the buoyancy and inertia of the air is $\tau q/g$ or $\tau \times 1{\cdot}25 \times 10^{-4}$.

(15) To this must be added the viscosity effect, which involves the absolute dimension of the pendulum.

It appears that for a cylindrical pendulum of the shape and dimensions assumed in these examples about 1/5 of the buoyancy inertia effect must be added to represent the frictional effect on the period.

(16) The other component of the viscosity effect acts on the amplitude and not on the period.

The total air effect therefore is to lengthen the period *in vacuo* by something like $1\frac{1}{2}$ parts in 10,000.

There is no complete theory for the viscous flow round short cylinders. Stokes' theory is for spheres and for very long cylinders, but the ordinary pendulum bob is too short to make the flow over the ends of the cylinder negligible, and the true correction must be greater than that proper to a sphere and less than for an infinitely long cylinder.

Owing to variations of the barometer and thermometer, the density of the air may at times differ from the mean by about 1/20 part; hence, from air effects alone, the rate of the assumed pendulum may change by about $7\frac{1}{2}$ parts in 1,000,000.

Escapements.

(17) In the dead-beat escapement it has been shown that the variation of rate is due to the variation of arc which must follow on variations of the coefficient of friction in the wheel work of the train, and, with regard to the latter, I have no data on which to form a calculation of a particular instance, but it appears almost certain, as may be seen by differentiating with regard to μ, the expression for the amplitude (on the supposition that f is proportional to the driving weight $\times (1-\mu)$) that the variation of amplitude is proportional to the variation of μ multiplied by some constant greater than unity.

(18) In the gravity escapement the variations of amplitude are small, but the variations of period due to the action of the escapement itself may amount to 10^{-6} for a variation of 1 per cent. in μ.

(19) In the model escapement friction does not enter, and the only source of variation of amplitude is the variation in the strength of the springs which may occur with variation of temperature and the possible variation of the phase at which the maintaining force is applied. Spring materials are now available, in which this alteration is almost insensible for such ranges of temperature as standard clocks are exposed to.

Compensation.

(20) Much thought and ingenuity has been given to methods of compensating pendulums for the effects of varying temperature.

The chief effect to be compensated is the variation of the length of the pendulum. If an uncompensated pendulum has a steel suspension rod, the variation of length is 0·000012 (about 1/80000) inch per degree centigrade. Hence, for each rise or fall of 1° C., the clock will lose or gain about half a second a day.

The alteration of length and of the moment of inertia of the bob is not the only source of variation of rate depending on temperature, for elasticity of the suspension spring and the density of the surrounding air are also involved, but, as all the variations are directly proportional to the change of temperature, any form of compensation which will correct for variation of length may with a slight alteration be made to correct for the rest.

All the compensations actually employed have depended, in one way or another, on the use of bodies with different coefficients of expansion. In practice, the pendulum rod is either a compound structure (generally zinc and steel) of parallel strips or bars joined at the ends, whose lengths are so adjusted that the large expansion of the zinc compensates the smaller expansion of the steel, or the bob is a cylindrical chamber containing mercury, whose dimensions are determined so as to make the expansion of the mercury do the necessary compensation.

The compound suspension rod is free from one objection to which the mercury chamber is open, namely, that the suspension rod takes the temperature of the surrounding air more quickly than the large mass of mercury, thus causing a lag in the adjustment of the compensation.

I have found that it is convenient to compensate the pendulum rod independently of the bob, and to attach the latter to the rod at such a distance below the centre of inertia that the variation of the length of the pendulum thus caused when the temperature varies is just balanced by the variation of the moment of inertia.

Thus, taking $(h^2+\kappa^2)/\lambda$ as the length of the equivalent pendulum, make the suspension rod of a length $h+x$, and let the attachment of the rod to the bob be made at x. The rod is compensated so that $h+x$ remains constant at all temperatures, and this compensation can be verified in the laboratory with great accuracy.

The distance of the centre of inertia of the bob from the point of suspension becomes, when the temperature changes by Δt, $h-x(1+a\Delta t)$ and the radius of gyration becomes $\kappa(1+a\Delta t)$, where a is the temperature

coefficient of the material of the bob, and, if we calculate x on the supposition that the equivalent length λ remains constant, we find $x = 2\kappa^2/h$. Thus x is independent of the coefficient of expansion of the material of the bob, and is a function only of its form.

With this arrangement the question of lag in compensation does not arise, at any rate in the first order of small quantities.

Materials (such as the alloy "invar" or melted quartz) are now available for suspension rods whose temperature coefficients are so small as to make special compensations for length unnecessary (the small outstanding expansions can be corrected by using a proper value for the x of the preceding paragraph), but these materials are not dense enough for advantageous use as pendulum bobs.

Airy first applied a compensation for the variation of the barometer. This was done by means of a magnet controlled by a float resting on the open mercury surface in the shorter leg of a syphon barometer. The movable magnet acted on another fixed to the pendulum, and, according to the height of the barometer, exercised a greater or less accelerating effect on the period. This compensation had to be adjusted by trial.

A simpler method is adopted in the model,* which has an additional advantage, in that it corrects for changes of density dependent on temperature as well as for pressure.

The whole pendulum and fittings are placed under a circular bell-glass, whose edge dips in a narrow, but deep, annular channel partly filled with mercury. Any change of density in the outer air causes the mercury to rise or fall on one side or other of the bell-glass, but, since the volume of the air within the glass is very large compared to the displacement caused by the movement of the mercury, the density of the air inside remains practically constant. The only variation of rate which the air can cause, except by variation of its density, is dependent on the change of viscosity with temperature, and this is small compared with the density effects.

In the particular case of the model, any change of external density is reduced to about 1/1000 within the bell-glass.

Considering the number of causes which may operate, especially in clocks with mechanical escapements, so as to vary the rate by quantities of the order of 1/1,000,000, it seems hardly likely that any device for maintaining the oscillation should ever secure a rate constant within 1/100,000,000. In particular, the maintenance of an arc of vibration constant to less than one second is hardly to be hoped for, for, even granting the maintaining force to be applied with perfection, the arc may vary in virtue of changes in

* The model referred to above (p. 517).

the stiffness, etc., of the support, that is, on outside circumstances, whose changes can neither be avoided nor compensated.

On the whole, I think it seems probable better time-keeping might be secured by a properly designed balance wheel oscillating on a spring than by a pendulum. The balance wheel and spring has the great advantage of something like real isochronism, so that small variations of arc are unimportant, and I believe there is no good reason to consider the elastic constants of a spring kept at a uniform temperature or otherwise compensated are more liable to variation than the force of gravity.

The Effect of Temperature upon the Ductility of Zinc.

By W. B. Haines, B.Sc., Physics Research Scholar, University College, London.

(Communicated by Prof. F. T. Trouton, F.R.S. Received June 7; revised July 22, 1911.)

It is well known that zinc has the property of becoming soft at temperatures in the region of 150° C., while it is said to become hard and brittle at about 200°. This applies to the commercial variety of the metal, which always contains a small proportion of lead. These changes suggested an investigation into the effect of temperature upon the ductile properties of zinc, of which the present paper is a record.

The experimental plan adopted was the examination of the ductility of zinc wire at various temperatures. Wires about 15 cm. long were fixed in a vertical position in a bath of castor oil, and stretched by a weight applied to the upper end by means of a pulley. A cathetometer was used to measure the extension of the wire. At each particular temperature the rate of extension of the wire was calculated from the observations for several loads ranging up to the breaking stress.

The results of the work emphasize the importance of treating the wires in exactly the same manner in order to gain consistent results. Thus the state of the wire as indicated by the rate of extension under a given stress depends not only upon the temperature, but also upon other factors, such as the previous treatment which the wire has received. In order to gain the required consistency of result, the wires were prepared and used in the order of their position on the original hank of wire, so minimising the effect of any variation in condition along the hank. An endeavour was also made